Cosmology

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Abstract

This Odyssey Project is an overview of General Relativity as well as Cosmology [2]. This poster begins by introducing General relativity followed by the Big Bang Model and conclude with problems with the Big Bang Model.

Differential Geometry

Our universe can be seen as a curved spacetime $\mathbb{R}^4$ (a 4-dimensional manifold) which contains integral curves. The tangent and cotangent spaces of the curves then form the vector spaces of tensors. Following which, the covariant derivative of vector W^μ are defined as (Christoffel symbols are used here):

$$\nabla_\nu W^\mu = \frac{\partial W^\mu}{\partial x^\nu} + \Gamma^\mu_{\nu\lambda} W^\lambda \tag{1}$$

Parallel transport of vectors can then be defined:

$$\boxed{\nabla_V W^\nu = 0 \Longrightarrow \frac{dW^\nu}{d\tau} + \Gamma^\nu_{\mu\sigma} W^\sigma \frac{dx^\mu}{d\tau}} \tag{2}$$

Geodesics on curved spacetime can then be found using the parallel transport of its tangent vector.

$$\boxed{\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\alpha} \frac{dx^\nu}{d\tau} \frac{dx^\alpha}{d\tau} = 0} \tag{3}$$

The intrinsic curvature of manifold can be found with parallel transport of a vector in a closed curve on the manifold to calculate the difference between the initial and final vector:

$$\boxed{R^\mu{}_{\rho\lambda\alpha} = \partial_\lambda \Gamma^\mu_{\alpha\rho} - \partial_\alpha \Gamma^\mu_{\lambda\rho} + \Gamma^\mu_{\lambda\beta}\Gamma^\beta_{\alpha\rho} - \Gamma^\mu_{\alpha\beta}\Gamma^\beta_{\lambda\rho}} \tag{4}$$

- Ricci Tensor: $R_{\mu\nu} = R^\sigma{}_{\mu\sigma\nu}$ symmetric in $\mu\nu$.
- Ricci Scalar: $R = R^\mu{}_\mu = g^{\mu\nu} R_{\mu\nu}$
- The contracted Bianchi identities:

$$\nabla^\beta \left(R_{\beta\delta} - \frac{1}{2} g_{\beta\delta} R \right) = 0 \tag{5}$$

General Relativity

The source of spacetime curvature is the Energy-momentum tensor.

$$T^{\mu\nu} = \begin{pmatrix} \substack{\text{energy}\\ \text{density}} & (\text{energy flux})_x & (\text{energy flux})_y & (\text{energy flux})_z \\ \binom{\text{momentum}}{\text{density}}_x & (\text{pressure})_x & \binom{\text{viscosity}}{\text{shear}}_{xy} & \binom{\text{viscosity}}{\text{shear}}_{xz} \\ \binom{\text{momentum}}{\text{density}}_y & \binom{\text{viscosity}}{\text{shear}}_{yx} & (\text{pressure})_y & \binom{\text{viscosity}}{\text{shear}}_{yz} \\ \binom{\text{momentum}}{\text{density}}_z & \binom{\text{viscosity}}{\text{shear}}_{zx} & \binom{\text{viscosity}}{\text{shear}}_{zy} & (\text{pressure})_z \end{pmatrix} \tag{6}$$

GR can be easily summarized in 2 main equations:

- Einstein's equation: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$ which dictates how spacetime $g_{\mu\nu}$ curves.
- Geodesic equation: $\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\alpha} \frac{dx^\nu}{d\tau}\frac{dx^\alpha}{d\tau} = 0$ which dictates the movement of matter.

The usual approach to GR is to assume some symmetries and partially fix the metric based on the symmetries, then substitute that metric into Einstein's equation and fix the rest of the parameters in the metric.

Friedmann Equations

Assuming galaxies are perfect fluid and our universe is spatially isotropic and homogeneous, we can derive the Friedmann Equations:

$$\boxed{\begin{cases} \ddot{a} = -\frac{4\pi G}{3} a \left(\rho_{(0)} + \frac{3p_{(0)}}{c^2} \right) + \frac{a}{3} \Lambda c^2 \\ \dot{a}^2 = -kc^2 + \frac{8\pi G}{3} \rho_{(0)} a^2 + \frac{1}{3} \Lambda a^2 c^2 \end{cases}}$$

The characteristics and evolution of the Universe can then be determined using these equations.

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Big Bang Cosmology(Friedmann models)

Dust-only Friedmann Models (so $\Omega_{r,0} = 0$ as well)

The rewritten 2nd friedmann equation gives

$$t = \frac{1}{H_0} \int_0^{\bar{a}} \sqrt{\frac{x}{\Omega_{m,0} + (1 - \Omega_{m,0})x}}\, dx \tag{7}$$

Special Case (a): For $\Omega_{m,0} = 1$, so $\Omega_{k,0} = 0$ and so $k = 0$:

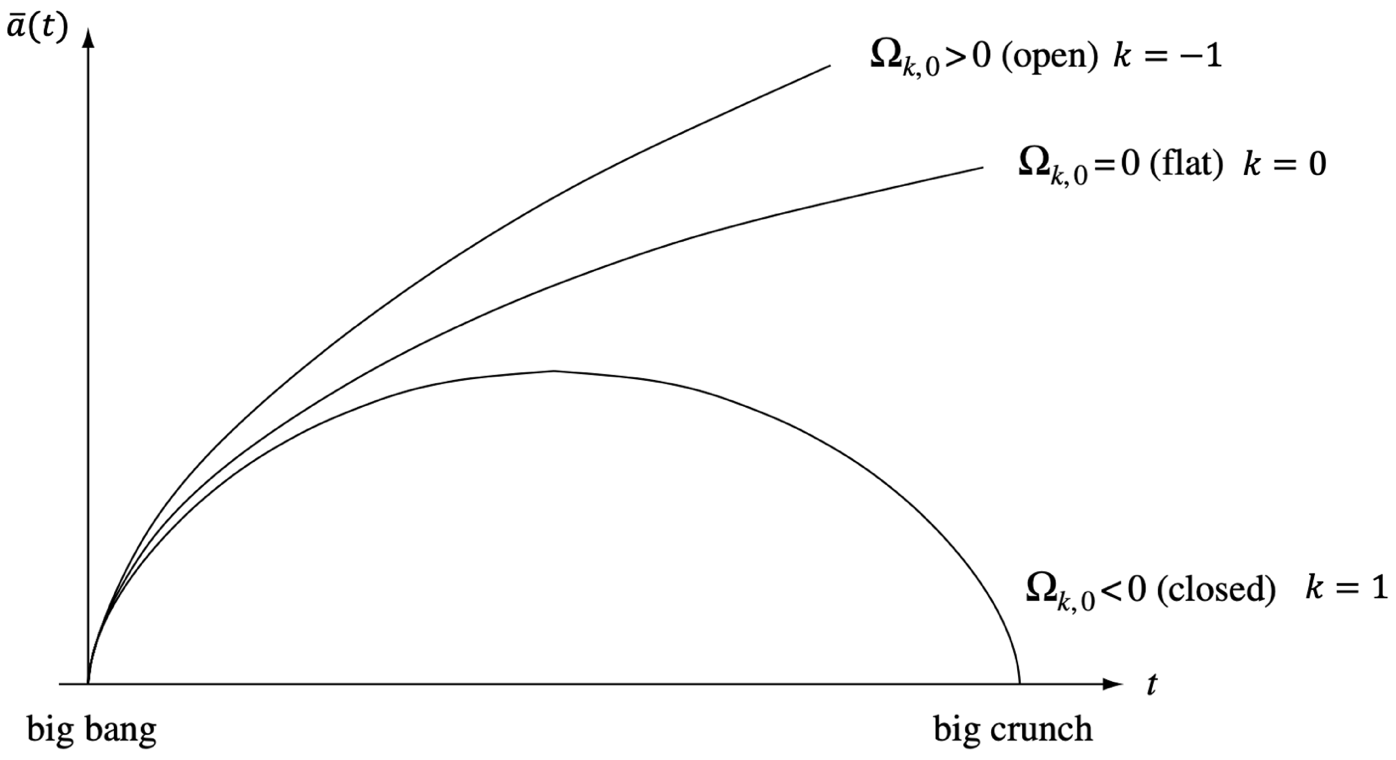
$$\bar{a} = \left(\frac{3}{2} H_0 t \right)^{2/3} \tag{8}$$

Special Case (b): For $\Omega_{m,0} > 1$ therefore $\Omega_{k,0} < 0$ and so $k = 1$:

$$\bar{a} = \frac{\Omega m, 0}{2(\Omega_{m,0} - 1)} (1 - \cos\psi) \tag{9}$$

Special Case (c): For $\Omega_{m,0} < 1$ therefore $\Omega_{k,0} > 0$ and so $k = -1$:

$$\bar{a} = \frac{\Omega_{m,0}}{2(1 - \Omega_{m,0})} (\cosh\psi - 1) \tag{10}$$



Problems with Big Bang Cosmology:

- Flatness Problem:

  The equation describing evolution of spatial curvature is

  $$\Omega_k(z) = \frac{\Omega_{k,0}}{\Omega_{m,0}(1+z) + \Omega_{r,0}(1+z)^2 + \Omega_{\Lambda,0}(1+z)^{-2} + \Omega_{k,0}}$$

  Since present day $\Omega_{k,0}$ is quite close to zero, it is implied that long ago Ω_k must be finely tuned to zero.

- Horizon Problem:

  Vastly separated regions have similar physical characteristics like having almost uniform microwave background radiation. However, these regions are outside each other's particle horizons and have no causal contact so they couldn't have interacted to achieve this uniformity in the temperature of CMB radiation.

References

[1] Michael Paul Hobson, George P Efstathiou, and Anthony N Lasenby. *General relativity: an introduction for physicists.* Cambridge University Press, 2006.

[2] Dr Leek Meng Lee. Notes for ph3403 cosmology, October 2020.

[3] Matts Roos. *Introduction to cosmology.* John Wiley & Sons, 2015.

[4] Bernard Schutz. *A First Course in General Relativity.* Cambridge University Press, 2 edition, 2009. doi: 10.1017/CBO9780511984181.